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Convergence of Approximations for Arrangements of Curves

Manuela Neagu and Bernard Lacolle

Abstract. Arrangements of planar objects represent one of the main topics of computational geometry. We propose an approach that allows reliable computations on arrangements of curves. This approach is based on the use of polygonal approximations for the curves composing the arrangement, and the questions to be answered concern the topological and geometric properties of the approximating arrangement of polygonal lines.

§1. Introduction

Problems on arrangements represent one of the most important topics in computational geometry. Arrangements find numerous applications, ranging from the design of 2D drawing tools [9] to motion planning, point location, and visibility problems [5].

Arrangements of hyperplanes, especially arrangements of lines in the plane, have been widely studied. Satisfactory theoretical results (e.g. the zone theorem [7]) and algorithmical results have been found. The interest is now focused on arrangements in higher dimensions, or on planar arrangements of objects other than lines: segments and (Jordan) curves, to give only a few examples. Dealing with arrangements of segments is more difficult than dealing with arrangements of lines because of the larger topological complexity of the cells. The case of curves raises also the fundamental problem of the computation with curves.

Most of the authors that have dealt with curve arrangements have adopted an approach based on the use of a small number of *oracles*. These oracles provide solutions for elementary geometric operations on curves and are considered acurately feasible in constant time. Examples of elementary operations solved by the oracles are the computation of the intersection of two curves (generally, of cardinality greater than 1), the computation of the vertical tangents to a curve, etc. This approach is adopted in papers presenting theoretical results rather than implemented algorithms and numerical results

[6,14,15]. Some papers presenting a more application-oriented approach can also be found in the literature [9,12].

In this paper, we present a new approach to the problems of arrangements of curves. We work in a strictly geometric framework, in which the only legal computations are those performed on linear objects. For this purpose, we shall use polygonal approximations of the curves defining the arrangement. Two important questions arise when such an approach is adopted. On the one hand, what kind of information can we obtain on the given arrangement of curves if we avoid algebraic equations? On the other hand, which are the restrictions we must impose on the arrangement of curves in order to assure that the required information can be provided by the arrangement of polygonal approximations?

On an arrangement, there are two different types of results. Firstly, there are topological (or combinatorial) characteristics expressed by the incidence graph. For example, this graph can be used to find the topological closure of a given cell or all its neighbours. Secondly, it can be useful to have geometric information on the faces of the arrangement. That would allow to make a decision for questions such as the point location problem. This paper briefly presents results for both aspects of curve arrangements.

We remark that if we want to compute the incidence graph of an arrangement of curves avoiding algebraic equations, then degenerate cases can not be treated. Indeed, if three curves have a common point, it generally cannot be found via polygonal approximations of the curves. Similarly, if two curves are tangent in a common point, algebraic equations must be generally used to detect the tangency. But on the other hand, our approach provides a robust algorithm for nondegenerate arrangements. Moreover, the method we propose detects the "almost" degenerate positions of the curves. If such a situation occurs, symbolic methods can be employed to obtain the exact local configuration of the arrangement.

The construction of the polygonal approximations is theoretically possible for Jordan curves as general as we want. Practically, the input of an algorithm should be more precise. We have thus chosen to deal with composite Bézier curves. A subject similar to the one of our paper, but concerning only mutually nonintersecting composite Bézier curves, has been treated in [2].

Every curve will be approximated by two polygonal lines: the *control polygon* and the *carrier polygon*. Our approach is thus similar to the one presented in [3,4].

The outline of our paper is the following. In Section 2, we introduce the polycurve, a composite Bézier curve satisfying certain conditions. This is the object we shall deal with throughout the paper. We also define the control and the carrier polygons of a polycurve.

In Section 3, we give basic definitions and notations concerning the simple arrangements of polycurves and control and carrier polygons. In Section 4, we deal with the equivalence of arrangements, for which the definition and sufficient conditions are given. We state the existence of an arrangement of control polygons equivalent to the one of polycurves.

In Section 5, we present our results on the convergence in terms of Hausdorff distance of the cells of the polygonal arrangements to the cells of the curve arrangement. In Section 6, we give a relation of inclusion between the faces of the three arrangements that allows a reliable approach for the point location problem. Finally, Section 7 concludes the paper.

§2. Polycurves

As we can see in the literature [6,12,14,15], the problems on arrangements can be addressed for curves which are subject to very few constraints. But the use of general curves makes the theoretical results unsuitable for direct implementation. This is one of the reasons why we restrict our study to piecewise *completely convex* Bézier curves:

Definition 1. *We say that a Bézier curve is* completely convex *if its control polygon is convex.*

We recall that a curve (or a polygonal chain) is called convex if it is simple and included in the boundary of its convex hull. A completely convex Bézier curve is obviously convex.

Definition 2. *A* polycurve *is a simple curve that can be written as a (finite) union of completely convex Bézier curves.*

An example of polycurve is presented in Figure 1. Let $\mathcal{B} = \cup_{i=1}^{n} \mathcal{B}_i$ be a polycurve, where the control polygon of the Bézier curve $\mathcal{B}_i$ is $\mathcal{P}_i = \mathrm{P}_0^{(i)} \mathrm{P}_1^{(i)} \ldots \mathrm{P}_{m_i}^{(i)}$.

Definition 3. *To any polycurve $\mathcal{B}$ we associate two polygonal chains:*
1) $\mathcal{P} = \cup_{i=1}^{n} \mathcal{P}_i$ *will be called the* bounding polygon *of $\mathcal{B}$;*
2) $\mathcal{S} = \cup_{i=1}^{n} [\mathrm{P}_0^{(i)} \mathrm{P}_{m_i}^{(i)}]$ *will be called the* carrier polygon *of $\mathcal{B}$.*

The bounding and carrier polygons are rough polygonal approximations of the corresponding polycurve. To refine these approximations, we shall apply successive de Casteljau subdivisions to the composing Bézier curves. The subdivision parameter is fixed and equal to 1/2. On the one hand, this value assures optimal (quadratic) convergence of the control polygon to the associate Bézier curve. On the other hand, the computations are easier and more accurate in this case. Indeed, if the subdivision parameter is equal to 1/2, then the only arithmetic operations required for the computation of the new control polygons are additions and divisions by 2.

Let us suppose that we subdivide the curve $\mathcal{B}_i$, obtaining the Bézier curves $\mathcal{B}_i'$ and $\mathcal{B}_i''$. Then

$$\mathcal{B}_i = \mathrm{B}_n\big(\mathrm{P}_0^{(i)}, \ldots, \mathrm{P}_{m_i}^{(i)}; [0,1]\big)$$
$$= \mathrm{B}_n\big(\mathrm{P'}_0^{(i)}, \ldots, \mathrm{P'}_{m_i}^{(i)}; [0,1]\big) \cup \mathrm{B}_n\big(\mathrm{P''}_0^{(i)}, \ldots, \mathrm{P''}_{m_i}^{(i)}; [0,1]\big) = \mathcal{B}_i' \cup \mathcal{B}_i''.$$

Thus, $\cup_{j=1}^{n} \mathcal{B}_j$ and $\big(\cup_{j=1}^{i-1} \mathcal{B}_j\big) \cup \mathcal{B}_i' \cup \mathcal{B}_i'' \cup \big(\cup_{j=i+1}^{n} \mathcal{B}_j\big)$ represent the same polycurve. The control and respectively carrier polygons of the two expressions
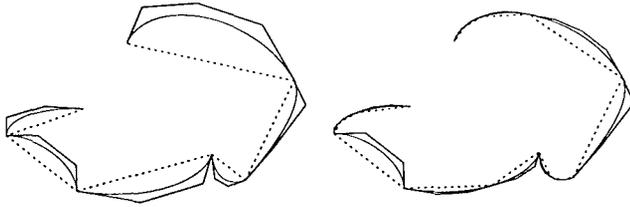
Fig. 1. A polycurve at two different levels of subdivision; the control and carrier
polygons are different.

are different: the two polygons associated with a polycurve are not unique,
and they change every time one of the composing Bézier curves is subdivided.
Figure 1 shows an example of a polycurve with its associate control and car-
rier polygons before and after the subdivision of some of the composing Bézier
curves.

§3. Arrangements of Curves

Let $\Gamma = \{C_i\}_{1 \leq i \leq n}$ be a set of Jordan curves. In this section we give the main
definitions related to the arrangement of the curves C_i.

3.1. Arrangement and incidence graph

Definition 4. *The arrangement $\mathcal{A}(\Gamma)$ is the planar subdivision induced by
the curves of Γ; that is, $\mathcal{A}(\Gamma)$ is a planar map whose vertices are the pairwise
intersection points of the curves of Γ, whose edges are maximal (open) con-
nected portions of the C_i's that do not contain a vertex, and whose faces are
the connected components of $\mathbb{R}^2 \setminus \Gamma$.*

The vertices, edges, and faces of an arrangement represent its cells of
dimension 0, 1, and 2, respectively.

Definition 5. *Let f and g be two cells of $\mathcal{A}(\Gamma)$. If the dimension of f is the
dimension of g plus 1 and g is on the boundary of f, we say that g is a subcell
of f (and f is a supercell of g). We also say that f and g define an incidence,
or are incident to one another.*

Using the previous definition, we can present a useful representation of
an arrangement, its incidence graph.

Definition 6. *The incidence graph of the arrangement $\mathcal{A}(\Gamma)$ is a graph $G =
(V, E)$ where there is a node in V for every cell of $\mathcal{A}(\Gamma)$, and an arc between
two nodes if the corresponding cells are incident.*

3.2. Simple arrangements

General arrangements of curves can present degeneracies making their study rather tricky. If three curves have a common point, this situation is more difficult to handle than the similar one in the case of arrangements of lines, due to the complexity of the description of the curves. Moreover, two curves can have a common point without crossing at that point (impossible for straight lines). The arrangement is then sensitive to small perturbations.

These are the reasons why, as most of the authors who have studied problems involving arrangements have also done, that we deal exclusively with simple arrangements.

Definition 7. *The arrangement* $\mathcal{A}(\mathcal{C})$ *is called* simple *if*

1) *the intersection of any three distinct curves* $\mathcal{C}_i$, $\mathcal{C}_j$, *and* $\mathcal{C}_k$ *is empty;*
2) *if two distinct curves* $\mathcal{C}_i$ *and* $\mathcal{C}_j$ *have common points, they cross transversely in each of these points;*
3) *the set* $\cup_{i=1}^{n} \mathcal{C}_i$ *is connected.*

§4. Topological Approximation

The first question that we answer is: Can we compute the incidence graph of an arrangement of polycurves dealing solely with the polygonal approximations of the polycurves? We have proven that if the arrangement of polycurves is simple, then the answer to this question is yes.

4.1. Equivalence of arrangements

Definition 8 (Grünbaum). *Let* $\mathcal{A}_1$ *and* A_2 *be two arrangements. We say that they are equivalent if there exists a bijection* $\varphi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ *such that if* f *and* g *define an incidence in* $\mathcal{A}_1$, *then* $\varphi(f)$ *and* $\varphi(g)$ *define an incidence in* A_2.

It is obvious that two arrangements are equivalent if and only if they have the same incidence graph. On the other hand, we remark that if $\mathcal{A}_1$ is simple and $\mathcal{A}_2$ is equivalent to it, this does not imply that A_2 is simple. Property 2 of Definition 7 is not preserved by the equivalence of arrangements.

Let $\mathcal{B} = \{\mathcal{B}_i\}_{1 \leq i \leq n}$ be a set of polycurves. The polycurve $\mathcal{B}_i$ is composed by n_i completely convex Bézier curves $\mathcal{B}_{i,j}$, $\mathcal{B}_i = \cup_{j=1}^{n_i} \mathcal{B}_{i,j}$. $\mathcal{B}_{i,j}$ has degree $m_{i,j}$, and its control polygon will be denoted by $\mathcal{P}_{i,j} = \mathrm{P}_0^{(i,j)} \mathrm{P}_1^{(i,j)} \ldots \mathrm{P}_{m_{i,j}}^{(i,j)}$. We are thus interested in the equivalence of $\mathcal{A}(\mathcal{B})$, $\mathcal{A}(\mathcal{P})$, and $\mathcal{A}(\mathcal{S})$.

Theorem 9. *Let us suppose that* $\mathcal{A}(\mathcal{B})$ *is simple.*
1) *We can obtain by de Casteljau subdivision a set* $\mathcal{P}$ *of control polygons and a set* $\mathcal{S}$ *of carrier polygons of the polycurves of* $\mathcal{B}$ *such that* $\mathcal{A}(\mathcal{P})$ *and* $\mathcal{A}(\mathcal{S})$ *are simple and they remain simple after any further subdivision of the Bézier curves composing the polycurves.*

2) *We can obtain by de Casteljau subdivision a set $\mathcal{P}$ of control polygons and a set $\mathcal{S}$ of carrier polygons of the polycurves of $\mathcal{B}$ such that $\mathcal{A}(\mathcal{P})$ and $\mathcal{A}(\mathcal{S})$ are equivalent to $\mathcal{A}(\mathcal{B})$ and they remain so after any further subdivision of the Bézier curves composing the polycurves.*

We do not present the proof of this theorem here. It is lengthy and presents no technical difficulty. It uses the geometric properties of Bézier curves, namely the variation diminishing property, the inclusion of the curves in the convex hull of its control polygon, and the convergence of the control (and carrier) polygon to the curve by de Casteljau subdivision.

4.2. Polygonal criteria of equivalence

Theorem 9 assures that we can obtain by de Casteljau subdivision arrangements of control, respectively carrier, polygons providing the incidence graph of the corresponding simple arrangement of polycurves. Once these two polygonal arrangements are obtained, the computation of the incidence graph of $\mathcal{A}(\mathcal{B})$ can be done by working solely with polygonal objects. We address now the problem of deciding whether the polygonal arrangements are equivalent to the curve arrangement by performing operations uniquely on linear objects.

Theorem 10. *If the number and the order of all the vertices of $\mathcal{A}(\mathcal{P})$ (respectively $\mathcal{A}(\mathcal{S})$) lying on $\mathcal{P}_i$ (respectively $\mathcal{S}_i$) are the same as the number and the order of all the vertices of $\mathcal{A}(\mathcal{B})$ lying on $\mathcal{B}_i$, for all $i \in \{1, \ldots, n\}$, then $\mathcal{A}(\mathcal{P})$ (respectively $\mathcal{A}(\mathcal{S})$) and $\mathcal{A}(\mathcal{B})$ are equivalent.*

Thanks to the geometric properties of Bézier curves, this theorem is a direct consequence of a result given by Vo Phi [15].

Lemma 11. *We can decide whether the hypotheses of Theorem 10 are fulfilled by dealing solely with the control and the carrier polygons of the polycurves.*

We have established sufficient conditions of equivalence on the two polygonal arrangements. We do not present them here, and just mention that there are two conditions. The first one assures that $\mathrm{card}(\mathcal{B}_{i_1,j_1} \cap \mathcal{B}_{i_2,j_2}) = 1$ when $\mathrm{card}(\mathcal{P}_{i_1,j_1} \cap \mathcal{P}_{i_2,j_2}) = \mathrm{card}(\mathcal{S}_{i_1,j_1} \cap \mathcal{S}_{i_2,j_2}) = 1$, for all $i_1 \neq i_2 \in \{1, \ldots, n\}$ and $j_k \in \{1, \ldots, n_{i_k}\}$, $k = 1, 2$. This implies the equality of the numbers of vertices respectively lying on $\mathcal{P}_i$, $\mathcal{B}_i$ and $\mathcal{S}_i$ in the corresponding arrangements, for all $i \in \{1, \ldots, n\}$. The second condition assures the good ordering of the vertices in the three arrangements.

§5. Approximation in Terms of Distance

Let $\mathcal{B} = \{\mathcal{B}_i\}_{1 \leq i \leq n}$ be a set of polycurves, and let us suppose that $\mathcal{A}(\mathcal{B})$, $\mathcal{A}(\mathcal{P})$, and $\mathcal{A}(\mathcal{S})$ are equivalent. We remark that the problem of the convergence in terms of distance can be addressed also if the three arrangements are not topologically identical, but the discussion is more complex in this case and we do not present it in this paper.
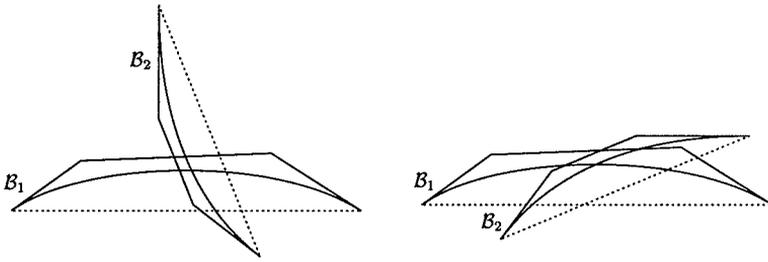
Fig. 2. Different angles between $\mathcal{B}_1$ and $\mathcal{B}_2$ at their intersection point.

5.1. Convergence of vertices

The proof of the following lemma is straightforward.

Lemma 12. *Any vertex of $\mathcal{A}(\mathcal{P})$ (respectively $\mathcal{A}(\mathcal{S})$) converges by subdivision to the corresponding vertex of $\mathcal{A}(\mathcal{B})$.*

We cannot give bounds on the distance between the corresponding vertices of $\mathcal{A}(\mathcal{B})$ and $\mathcal{A}(\mathcal{P})$ (respectively $\mathcal{A}(\mathcal{S})$) depending solely on the Hausdorff distance between the Bézier curves and their control (respectively carrier) polygons. Indeed, it is easy to see that the distance between the intersection point of the curves and the intersection point of the control (respectively carrier) polygons depends on the angle between the curves. An example is presented in Figure 2.

5.2. Convergence of edges

The proof of the following lemma is also straightforward.

Lemma 13. *Let $e_\mathcal{B}$ and $e_\mathcal{P}$ be corresponding edges of $\mathcal{A}(\mathcal{B})$ and $\mathcal{A}(\mathcal{P})$, and let $v_\mathcal{B}, v'_\mathcal{B}$ and $v_\mathcal{P}, v'_\mathcal{P}$ be their corresponding endpoints. There exists $i \in \{1, \ldots, n\}$ and $j_1, \ell_1 \in \{1, \ldots, m_i\}$, $j_1 \le \ell_1$, such that $e_\mathcal{B} \subset \cup_{k=j_1}^{\ell_1} \mathcal{B}_k^{(i)}$. Then*

$$\delta^{\mathrm{H}}(e_\mathcal{B}, e_\mathcal{P}) \le \max \left\{ \mathrm{d}(v_\mathcal{B}, v_\mathcal{P}), \mathrm{d}(v'_\mathcal{B}, v'_\mathcal{P}), \max_{k=j_1}^{\ell_1} \delta^{\mathrm{H}} \left(\mathcal{B}_k^{(i)}, \mathcal{P}_k^{(i)} \right) \right\}.$$

It is obvious that a similar relation holds for the edges of the arrangement of carrier polygons. Lemmas 12 and 13 immediately imply the following statement:

Corollary 14. *Any edge of $\mathcal{A}(\mathcal{P})$ (respectively $\mathcal{A}(\mathcal{S})$) converges by subdivision to the corresponding edge of $\mathcal{A}(\mathcal{B})$.*

5.3. Convergence of faces

The proof of the following lemma is straightforward.

Lemma 15. *Let $c_\mathcal{P}$ and $c_\mathcal{B}$ be corresponding faces of $\mathcal{A}(\mathcal{P})$ and $\mathcal{A}(\mathcal{B})$. Then*

$$\delta^{\mathrm{H}}(c_\mathcal{P}, c_\mathcal{B}) \leq \delta^{\mathrm{H}}\left(\delta(c_\mathcal{P}), \delta(c_\mathcal{B})\right),$$

where $\delta(A)$ denotes the boundary of the set A.

As a matter of fact, this is a more general result, holding for any two compact sets in the plane. Thus, this property is fulfilled also by the faces of the arangement $\mathcal{A}(\mathcal{S})$.

The convergence of faces is an immediate consequence of Lemma 12, Corollary 14, and Lemma 15.

Corollary 16. *Any face of $\mathcal{A}(\mathcal{P})$ (respectively $\mathcal{A}(\mathcal{S})$) converges by subdivision to the corresponding face of $\mathcal{A}(\mathcal{B})$.*

5.4. Polygonal criteria for the convergence in terms of distance

As in Theorem 9 on toplogical convergence of the polygonal arrangements to the curve arrangement, the results we have presented so far in this section imply computations with curves. We give here two results which allow us to estimate the Hausdorff distance between the corresponding cells of $\mathcal{A}(\mathcal{B})$ and $\mathcal{A}(\mathcal{B})$ (respectively $\mathcal{A}(\mathcal{S})$) by performing computations uniquely on polygonal lines. The proofs of the following lemmas are straightforward.

Lemma 17. *If $\mathcal{B}$ is a completely convex Bézier curve and $\mathcal{P} = \mathrm{P}_0\mathrm{P}_1\ldots\mathrm{P}_m$ is its control polygon, then*

$$\delta^{\mathrm{H}}(\mathcal{P}, \mathcal{B}) \leq \delta^{\mathrm{H}}\left(\mathcal{P}, [\mathrm{P}_0\mathrm{P}_m]\right), \qquad \delta^{\mathrm{H}}\left([\mathrm{P}_0\mathrm{P}_m], \mathcal{B}\right) \leq \delta^{\mathrm{H}}\left(\mathcal{P}, [\mathrm{P}_0\mathrm{P}_m]\right).$$

Lemma 18. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two completely convex Bézier curves and $\mathcal{P}_i = \mathrm{P}_0^{(i)}\mathrm{P}_1^{(i)}\ldots\mathrm{P}_{m_i}^{(i)}$, $i = 1, 2$, be respectively their control polygons. We suppose that*

$$\mathrm{card}\left(\mathcal{P}_1 \cap \mathcal{P}_2\right) = \mathrm{card}\left(\mathcal{B}_1 \cap \mathcal{B}_2\right) = \mathrm{card}\left([\mathrm{P}_0^{(1)}\mathrm{P}_{m_1}^{(1)}] \cap [\mathrm{P}_0^{(2)}\mathrm{P}_{m_2}^{(2)}]\right) = 1.$$

Then

$$\mathrm{d}\left(\mathcal{P}_1 \cap \mathcal{P}_2, \mathcal{B}_1 \cap \mathcal{B}_2\right) \leq \mathrm{d}\left(\mathcal{P}_1 \cap \mathcal{P}_2, [\mathrm{P}_0^{(1)}\mathrm{P}_{m_1}^{(1)}] \cap [\mathrm{P}_0^{(2)}\mathrm{P}_{m_2}^{(2)}]\right)$$

and

$$\mathrm{d}\left([\mathrm{P}_0^{(1)}\mathrm{P}_{m_1}^{(1)}] \cap [\mathrm{P}_0^{(2)}\mathrm{P}_{m_2}^{(2)}], \mathcal{B}_1 \cap \mathcal{B}_2\right) \leq \mathrm{d}\left(\mathcal{P}_1 \cap \mathcal{P}_2, [\mathrm{P}_0^{(1)}\mathrm{P}_{m_1}^{(1)}] \cap [\mathrm{P}_0^{(2)}\mathrm{P}_{m_2}^{(2)}]\right).$$
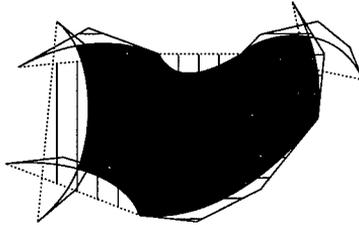
Fig. 3. $c_\mathcal{B}$ is not included in $c_\mathcal{P} \cap c_\mathcal{S}$.

§6. Inclusion of Faces

The results given in Section 5 can be very useful for solving the point location problem in the case of curve arrangements. In this section, we present a result that shows how the convergence in terms of distance of the faces of $\mathcal{A}(\mathcal{P})$ and respectively $\mathcal{A}(\mathcal{S})$ to the faces of $\mathcal{A}(\mathcal{B})$ is applied to the mentioned problem. We use for the set of polycurves $\mathcal{B}$ the notations of Section 4, that is, $\mathcal{B} = \{\mathcal{B}_i\}_{1 \leq i \leq n}$, where $\mathcal{B}_i = \cup_{j=1}^{n_i} \mathcal{B}_{i,j}$ with $\mathcal{B}_{i,j}$ completely convex Bézier curves for all i and j.

Lemma 19. *Let $c_\mathcal{B}$ be a face of $\mathcal{A}(\mathcal{B})$, and $c_\mathcal{P}$, $c_\mathcal{S}$ be the faces that correspond to $c_\mathcal{B}$ in $\mathcal{A}(\mathcal{P})$, respectively $\mathcal{A}(\mathcal{S})$. Let $v_1, \ldots, v_p$ be all the vertices of $\mathcal{A}(\mathcal{B})$ lying on the boundary of $c_\mathcal{B}$. There exist $i_k \neq i'_k \in \{1, \ldots, n\}$ and $j_k \in \{1, \ldots, n_i\}$, $j'_k \in \{1, \ldots, n_{i'}\}$, for all $k \in \{1, \ldots, p\}$, such that $v_k = \mathcal{B}_{i_k, j_k} \cap \mathcal{B}_{i'_k, j'_k}$. Then*

$$c_\mathcal{P} \cap c_\mathcal{S} \subset c_\mathcal{B} \subset c_\mathcal{P} \cup c_\mathcal{S} \cup \bigcup_{k=1}^{p} \left(reg(\mathcal{P}_{i_k, j_k}) \cap reg(\mathcal{P}_{i'_k, j'_k}) \right),$$

where $reg(\mathcal{P}_{i,j})$ denotes the bounded region enclosed by the polygon $\mathcal{P}_{i,j}$.

This property is illustrated in Figure 3. We remark that in fact the terms $reg(\mathcal{P}_{i_k, j_k}) \cap reg(\mathcal{P}_{i'_k, j'_k})$ are not all necessary. When the Bézier curves $\mathcal{B}_{i_k, j_k}$ and $\mathcal{B}_{i'_k, j'_k}$ both have their "convex side" oriented either to the interior of the face $c_\mathcal{B}$ or to the exterior of this face, we do not have to add $reg(\mathcal{P}_{i_k, j_k}) \cap reg(\mathcal{P}_{i'_k, j'_k})$ to the union above.

§7. Conclusion

In this paper, we briefly present results concerning the use of polygonal approximations for solving important computational geometry problems on arrangements of curves. We have dealt with two different types of problems, topological and geometric. For both kinds of problems, the polygonal approximations represent a suitable tool, providing solutions that do not require solving algebraic equations.

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